and $p(X_3 = 0) = 1/2$, so it is true that $p(X_1 = 0 \wedge X_3 = 0) = p(X_1 = 0)p(X_3 = 0)$. Essentially the same calculation shows that $p(X_1 = 0 \wedge X_3 = 1) = p(X_1 = 0)p(X_3 = 1)$, $p(X_1 = 1 \wedge X_3 = 0) = p(X_1 = 1)p(X_3 = 0)$, and $p(X_1 = 1 \wedge X_3 = 1) = p(X_1 = 1)p(X_3 = 1)$. Therefore by definition, X_1 and X_3 are independent. The same reasoning shows that X_2 and X_3 are independent. To see that X_3 and $X_1 + X_2$ are not independent, we observe that $p(X_3 = 1 \wedge X_1 + X_2 = 2) = 0$. But $p(X_3 = 1)p(X_1 + X_2 = 2) = (1/2)(1/4) = 1/8$. **b)** We see from the calculation in part (a) that X_1, X_2, and X_3 are all Bernoulli random variables, so the variance of each is $(1/2)(1/2) = 1/4$. Therefore, $V(X_1) + V(X_2) + V(X_3) = 3/4$. We use the calculations in part (a) to see that $E(X_1 + X_2 + X_3) = 3/2$, and then $V(X_1 + X_2 + X_3) = 3/4$. **c)** In order to use the first part of Theorem 7 to show that $V((X_1 + X_2 + \cdots + X_k) + X_{k+1}) = V(X_1 + X_2 + \cdots + X_k) + V(X_{k+1})$ in the inductive step of a proof by mathematical induction, we would have to know that $X_1 + X_2 + \cdots + X_k$ and X_{k+1} are independent, but we see from part (a) that this is not necessarily true. **29.** $1/100$ **31.** $E(X)/a = \sum_r (r/a) \cdot p(X = r) \geq \sum_{r \geq a} 1 \cdot p(X = r) = p(X \geq a)$ **33. a)** $10/11$ **b)** 0.9984 **35. a)** Each of the $n!$ permutations occurs with probability $1/n!$, so $E(X)$ is the number of comparisons, averaged over all these permutations. **b)** Even if the algorithm continues $n - 1$ rounds, X will be at most $n(n-1)/2$. It follows from the formula for expectation that $E(X) \leq n(n-1)/2$. **c)** The algorithm proceeds by comparing adjacent elements and then swapping them if necessary. Thus, the only way that inverted elements can become uninverted is for them to be compared and swapped. **d)** Because $X(P) \geq I(P)$ for all P, it follows from the definition of expectation that $E(X) \geq E(I)$. **e)** This summation counts 1 for every instance of an inversion. **f)** This follows from Theorem 3. **g)** By Theorem 2 with $n = 1$, the expectation of $I_{j,k}$ is the probability that a_k precedes a_j in the permutation. This is clearly 1/2 by symmetry. **h)** The summation in part (f) consists of $C(n, 2) = n(n-1)/2$ terms, each equal to 1/2, so the sum is $n(n-1)/4$. **i)** From part (a) and part (b) we know that $E(X)$, the object of interest, is at most $n(n-1)/2$, and from part (d) and part (h) we know that $E(X)$ is at least $n(n-1)/4$, both of which are $\Theta(n^2)$. **37.** 1 **39.** $V(X + Y) = E((X+Y)^2) - E(X+Y)^2 = E(X^2 + 2XY + Y^2) - [E(X) + E(Y)]^2 = E(X^2) + 2E(XY) + E(Y^2) - E(X)^2 - 2E(X)E(Y) - E(Y)^2 = E(X^2) - E(X)^2 + 2[E(XY) - E(X)E(Y)] + E(Y^2) - E(Y)^2 = V(X) + 2\,\mathrm{Cov}(X, Y) + V(Y)$ **41.** $[(n-1)/n]^m$ **43.** $(n-1)^m/n^{m-1}$

Supplementary Exercises

1. $1/109{,}668$ **3. a)** $1/C(52, 13)$ **b)** $4/C(52, 13)$ **c)** $2{,}944{,}656/C(52, 13)$ **d)** $35{,}335{,}872/C(52, 13)$ **5. a)** $9/2$ **b)** $21/4$ **7. a)** 9 **b)** $21/2$ **9. a)** 8 **b)** $49/6$ **11. a)** $n/2^{n-1}$ **b)** $p(1-p)^{k-1}$, where $p = n/2^{n-1}$ **c)** $2^{n-1}/n$ **13.** $\frac{(m-1)(n-1)+\gcd(m,n)-1}{mn-1}$ **15. a)** $2/3$ **b)** $2/3$ **17.** $1/32$ **19. a)** The probability that one wins 2^n dollars is $1/2^n$, because that happens precisely when the player gets $n - 1$ tails followed by a head. The expected value of the winnings is therefore the sum of 2^n times $1/2^n$ as n goes from 1 to infinity. Because each of these terms is 1, the sum is infinite. In other words, one should be willing to wager any amount of money and expect to come out ahead in the long run. **b)** \$9, \$9 **21. a)** $1/3$ when $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and $B = \{1, 2, 3, 4\}$; $1/12$ when $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $A = \{4, 5, 6, 7, 8, 9, 10, 11, 12\}$, and $B = \{1, 2, 3, 4\}$ **b)** 1 when $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $A = \{4, 5, 6, 7, 8, 9, 10, 11, 12\}$, and $B = \{1, 2, 3, 4\}$; 3/4 when $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and $B = \{1, 2, 3, 4\}$ **23. a)** $p(E_1 \cap E_2) = p(E_1)p(E_2)$, $p(E_1 \cap E_3) = p(E_1)p(E_3)$, $p(E_2 \cap E_3) = p(E_2)p(E_3)$, $p(E_1 \cap E_2 \cap E_3) = p(E_1)p(E_2)p(E_3)$ **b)** Yes **c)** Yes; yes **d)** Yes; no **e)** $2^n - n - 1$ **25.** $V(aX + b) = E((aX + b)^2) - E(aX + b)^2 = E(a^2X^2 + 2abX + b^2) - [aE(X) + b]^2 = E(a^2X^2) + E(2abX) + E(b^2) - [a^2E(X)^2 + 2abE(X) + b^2] = a^2E(X^2) + 2abE(X) + b^2 - a^2E(X)^2 - 2abE(X) - b^2 = a^2[E(X^2) - E(X)^2] = a^2V(X)$ **27.** To count every element in the sample space exactly once, we must include every element in each of the sets and then take away the double counting of the elements in the intersections. Thus $p(E_1 \cup E_2 \cup \cdots \cup E_m) = p(E_1) + p(E_2) + \cdots + p(E_m) - p(E_1 \cap E_2) - p(E_1 \cap E_3) - \cdots - p(E_1 \cap E_m) - p(E_2 \cap E_3) - p(E_2 \cap E_4) - \cdots - p(E_2 \cap E_m) - \cdots - p(E_{m-1} \cap E_m) = qm - (m(m-1)/2)r$, because $C(m, 2)$ terms are being subtracted. But $p(E_1 \cup E_2 \cup \cdots \cup E_m) = 1$, so we have $qm - [m(m-1)/2]r = 1$. Because $r \geq 0$, this equation tells us that $qm \geq 1$, so $q \geq 1/m$. Because $q \leq 1$, this equation also implies that $[m(m-1)/2]r = qm - 1 \leq m - 1$, from which it follows that $r \leq 2/m$. **29. a)** We purchase the cards until we have gotten one of each type. That means we have purchased X cards in all. On the other hand, that also means that we purchased X_0 cards until we got the first type we got, and then purchased X_1 more cards until we got the second type we got, and so on. Thus, X is the sum of the X_j's. **b)** Once j distinct types have been obtained, there are $n - j$ new types available out of a total of n types available. Because it is equally likely that we get each type, the probability of success on the next purchase (getting a new type) is $(n-j)/n$. **c)** This follows immediately from the definition of geometric distribution, the definition of X_j, and part (b). **d)** From part (c) it follows that $E(X_j) = n/(n-j)$. Thus by the linearity of expectation and part (a), we have $E(X) = E(X_0) + E(X_1) + \cdots + E(X_{n-1}) = \frac{n}{n} + \frac{n}{n-1} + \cdots + \frac{n}{1} = n\left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{1}\right)$. **e)** About 224.46 **31.** $24 \cdot 13^4/(52 \cdot 51 \cdot 50 \cdot 49)$

CHAPTER 7

Section 7.1

1. a) 2, 12, 72, 432, 2592 **b)** 2, 4, 16, 256, 65,536 **c)** 1, 2, 5, 11, 26 **d)** 1, 1, 6, 27, 204 **e)** 1, 2, 0, 1, 3

3. a) 6, 17, 49, 143, 421 **b)** $49 = 5 \cdot 17 - 6 \cdot 6$, $143 = 5 \cdot 49 - 6 \cdot 17$, $421 = 5 \cdot 143 - 6 \cdot 49$ **c)** $5a_{n-1} - 6a_{n-2} = 5(2^{n-1} + 5 \cdot 3^{n-1}) - 6(2^{n-2} + 5 \cdot 3^{n-2}) = 2^{n-2}(10-6) + 3^{n-2}(75-30) = 2^{n-2} \cdot 4 + 3^{n-2} \cdot 9 \cdot 5 = 2^n + 3^n \cdot 5 = a_n$ **5. a)** Yes **b)** No **c)** No **d)** Yes **e)** Yes **f)** Yes **g)** No **h)** No **7. a)** $a_{n-1} + 2a_{n-2} + 2n - 9 = -(n-1) + 2 + 2[-(n-2)+2] + 2n - 9 = -n + 2 = a_n$ **b)** $a_{n-1} + 2a_{n-2} + 2n - 9 = 5(-1)^{n-1} - (n-1) + 2 + 2[5(-1)^{n-2} - (n-2) + 2] + 2n - 9 = 5(-1)^{n-2}(-1+2) - n + 2 = a_n$ **c)** $a_{n-1} + 2a_{n-2} + 2n - 9 = 3(-1)^{n-1} + 2^{n-1} - (n-1) + 2 + 2[3(-1)^{n-2} + 2^{n-2} - (n-2) + 2] + 2n - 9 = 3(-1)^{n-2}(-1+2) + 2^{n-2}(2+2) - n + 2 = a_n$ **d)** $a_{n-1} + 2a_{n-2} + 2n - 9 = 7 \cdot 2^{n-1} - (n-1) + 2 + 2[7 \cdot 2^{n-2} - (n-2) + 2] + 2n - 9 = 2^{n-2}(7 \cdot 2 + 2 \cdot 7) - n + 2 = a_n$ **9. a)** $a_n = 2 \cdot 3^n$ **b)** $a_n = 2n + 3$ **c)** $a_n = 1 + n(n+1)/2$ **d)** $a_n = n^2 + 4n + 4$ **e)** $a_n = 1$ **f)** $a_n = (3^{n+1} - 1)/2$ **g)** $a_n = 5n!$ **h)** $a_n = 2^n n!$ **11. a)** $a_n = 3a_{n-1}$ **b)** 5,904,900 **13. a)** $a_n = n + a_{n-1}$, $a_0 = 0$ **b)** $a_{12} = 78$ **c)** $a_n = n(n+1)/2$ **15.** $B(k) = [1 + (0.07/12)]B(k-1) - 100$, with $B(0) = 5000$ **17.** Let $P(n)$ be "$H_n = 2^n - 1$." *Basis step:* $P(1)$ is true because $H_1 = 1$. *Inductive step:* Assume that $H_n = 2^n - 1$. Then because $H_{n+1} = 2H_n + 1$, it follows that $H_{n+1} = 2(2^n - 1) + 1 = 2^{n+1} - 1$. **19. a)** $a_n = 2a_{n-1} + a_{n-5}$ for $n \geq 5$ **b)** $a_0 = 1$, $a_1 = 2$, $a_3 = 8$, $a_4 = 16$ **c)** 1217 **21.** 9494 **23. a)** $a_n = a_{n-1} + a_{n-2} + 2^{n-2}$ for $n \geq 2$ **b)** $a_0 = 0$, $a_1 = 0$ **c)** 94 **25. a)** $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 3$ **b)** $a_0 = 1$, $a_1 = 2$, $a_2 = 4$ **c)** 81 **27. a)** $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$ **b)** $a_0 = 1$, $a_1 = 1$ **c)** 34 **29. a)** $a_n = 2a_{n-1} + 2a_{n-2}$ for $n \geq 2$ **b)** $a_0 = 1$, $a_1 = 3$ **c)** 448 **31. a)** $a_n = 2a_{n-1} + a_{n-2}$ for $n \geq 2$ **b)** $a_0 = 1$, $a_1 = 3$ **c)** 239 **33. a)** $a_n = 2a_{n-1}$ for $n \geq 2$ **b)** $a_1 = 3$ **c)** 96 **35. a)** $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$ **b)** $a_0 = 1$, $a_1 = 1$ **c)** 89 **37. a)** $R_n = n + R_{n-1}$, $R_0 = 1$ **b)** $R_n = n(n+1)/2 + 1$ **39. a)** $S_n = S_{n-1} + (n^2 - n + 2)/2$, $S_0 = 1$ **b)** $S_n = (n^3 + 5n + 6)/6$ **41.** 64 **43. a)** $a_n = 2a_{n-1} + 2a_{n-2}$ **b)** $a_0 = 1$, $a_1 = 3$ **c)** 1224 **45.** Clearly, $S(m, 1) = 1$ for $m \geq 1$. If $m \geq n$, then a function that is not onto from the set with m elements to the set with n elements can be specified by picking the size of the range, which is an integer between 1 and $n - 1$ inclusive, picking the elements of the range, which can be done in $C(n, k)$ ways, and picking an onto function onto the range, which can be done in $S(m, k)$ ways. Hence, there are $\sum_{k=1}^{n-1} C(n,k)S(m,k)$ functions that are not onto. But there are n^m functions altogether, so $S(m,n) = n^m - \sum_{k=1}^{n-1} C(n,k)S(m,k)$. **47. a)** $C_5 = C_0C_4 + C_1C_3 + C_2C_2 + C_3C_1 + C_4C_0 = 1 \cdot 14 + 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 + 14 \cdot 1 = 42$ **b)** $C(10, 5)/6 = 42$ **49.** $J(1) = 1$, $J(2) = 1$, $J(3) = 3$, $J(4) = 1$, $J(5) = 3$, $J(6) = 5$, $J(7) = 7$, $J(8) = 1$, $J(9) = 3$, $J(10) = 5$, $J(11) = 7$, $J(12) = 9$, $J(13) = 11$, $J(14) = 13$, $J(15) = 15$, $J(16) = 1$ **51.** First, suppose that the number of people is even, say $2n$. After going around the circle once and returning to the first person, because the people at locations with even numbers have been eliminated, there are exactly n people left and the person currently at location i is the person who was originally at location $2i - 1$. Therefore, the survivor [originally in location $J(2n)$] is now in location $J(n)$; this was the person who was at location $2J(n) - 1$. Hence, $J(2n) = 2J(n) - 1$. Similarly, when there are an odd number of people, say $2n + 1$, then after going around the circle once and then eliminating person 1, there are n people left and the person currently at location i is the person who was at location $2i + 1$. Therefore, the survivor will be the player currently occupying location $J(n)$, namely, the person who was originally at location $2J(n) + 1$. Hence, $J(2n+1) = 2J(n) + 1$. The base case is $J(1) = 1$. **53.** 73, 977, 3617 **55.** These nine moves solve the puzzle: Move disk 1 from peg 1 to peg 2; move disk 2 from peg 1 to peg 3; move disk 1 from peg 2 to peg 3; move disk 3 from peg 1 to peg 2; move disk 4 from peg 1 to peg 4; move disk 3 from peg 2 to peg 4; move disk 1 from peg 3 to peg 2; move disk 2 from peg 3 to peg 4; move disk 1 from peg 2 to peg 4. To see that at least nine moves are required, first note that at least seven moves are required no matter how many pegs are present: three to unstack the disks, one to move the largest disk 4, and three more moves to restack them. At least two other moves are needed, because to move disk 4 from peg 1 to peg 4 the other three disks must be on pegs 2 and 3, so at least one move is needed to restack them and one move to unstack them. **57.** The base cases are obvious. If $n > 1$, the algorithm consists of three stages. In the first stage, by the inductive hypothesis, $R(n-k)$ moves are used to transfer the smallest $n - k$ disks to peg 2. Then using the usual three-peg Tower of Hanoi algorithm, it takes $2^k - 1$ moves to transfer the rest of the disks (the largest k disks) to peg 4, avoiding peg 2. Then again by the inductive hypothesis, it takes $R(n-k)$ moves to transfer the smallest $n - k$ disks to peg 4; all the pegs are available for this, because the largest disks, now on peg 4, do not interfere. This establishes the recurrence relation. **59.** First note that $R(n) = \sum_{j=1}^{n}[R(j) - R(j-1)]$ [which follows because the sum is telescoping and $R(0) = 0$]. By Exercise 58, this is the sum of $2^{k'-1}$ for this range of values of j. Therefore, the sum is $\sum_{i=1}^{k} i2^{i-1}$, except that if n is not a triangular number, then the last few values when $i = k$ are missing, and that is what the final term in the given expression accounts for. **61.** By Exercise 59, $R(n)$ is no larger than $\sum_{i=1}^{k} i2^{i-1}$. It can be shown that this sum equals $(k+1)2^k - 2^{k+1} + 1$, so it is no greater than $(k+1)2^k$. Because $n > k(k-1)/2$, the quadratic formula can be used to show that $k < 1 + \sqrt{2n}$ for all $n > 1$. Therefore, $R(n)$ is bounded above by $(1 + \sqrt{2n} + 1)2^{1+\sqrt{2n}} < 8\sqrt{n}2^{\sqrt{2n}}$ for all $n > 2$. Hence, $R(n)$ is $O(\sqrt{n}2^{\sqrt{2n}})$. **63. a)** 0 **b)** 0 **c)** 2 **d)** $2^{n-1} - 2^{n-2}$ **65.** $a_n - 2\nabla a_n + \nabla^2 a_n = a_n - 2(a_n - a_{n-1}) + (\nabla a_n - \nabla a_{n-1}) = -a_n + 2a_{n-1} + [(a_n - a_{n-1}) - (a_{n-1} - a_{n-2})] = -a_n + 2a_{n-1} + (a_n - 2a_{n-1} + a_{n-2}) = a_{n-2}$ **67.** $a_n = a_{n-1} + a_{n-2} = (a_n - \nabla a_n) + (a_n - 2\nabla a_n + \nabla^2 a_n) = 2a_n - 3\nabla a_n + \nabla^2 a_n$, or $a_n = 3\nabla a_n - \nabla^2 a_n$

Section 7.2

1. a) Degree 3 **b)** No **c)** Degree 4 **d)** No **e)** No **f)** Degree 2 **g)** No **3. a)** $a_n = 3 \cdot 2^n$ **b)** $a_n = 2$ **c)** $a_n =$

$3 \cdot 2^n - 2 \cdot 3^n$ **d)** $a_n = 6 \cdot 2^n - 2 \cdot n2^n$ **e)** $a_n = n(-2)^{n-1}$ **f)** $a_n = 2^n - (-2)^n$ **g)** $a_n = (1/2)^{n+1} - (-1/2)^{n+1}$ **5.** $a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$ **7.** $[2^{n+1} + (-1)^n]/3$ **9. a)** $P_n = 1.2P_{n-1} + 0.45P_{n-2}$, $P_0 = 100{,}000$, $P_1 = 120{,}000$ **b)** $P_n = (250{,}000/3)(3/2)^n + (50{,}000/3)(-3/10)^n$ **11. a)** *Basis step:* For $n = 1$ we have $1 = 0 + 1$, and for $n = 2$ we have $3 = 1 + 2$. *Inductive step:* Assume true for $k \le n$. Then $L_{n+1} = L_n + L_{n-1} = f_{n-1} + f_{n+1} + f_{n-2} + f_n = (f_{n-1} + f_{n-2}) + (f_{n+1} + f_n) = f_n + f_{n+2}$. **b)** $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$ **13.** $a_n = 8(-1)^n - 3(-2)^n + 4 \cdot 3^n$ **15.** $a_n = 5 + 3(-2)^n - 3^n$ **17.** Let $a_n = C(n, 0) + C(n-1, 1) + \cdots + C(n-k, k)$ where $k = \lfloor n/2 \rfloor$. First, assume that n is even, so that $k = n/2$, and the last term is $C(k, k)$. By Pascal's Identity we have $a_n = 1 + C(n-2, 0) + C(n-2, 1) + C(n-3, 1) + C(n-3, 2) + \cdots + C(n-k, k-2) + C(n-k, k-1) + 1 = 1 + C(n-2, 1) + C(n-3, 2) + \cdots + C(n-k, k-1) + C(n-2, 0) + C(n-3, 1) + \cdots + C(n-k, k-2) + 1 = a_{n-1} + a_{n-2}$ because $\lfloor (n-1)/2 \rfloor = k - 1 = \lfloor (n-2)/2 \rfloor$. A similar calculation works when n is odd. Hence, $\{a_n\}$ satisfies the recurrence relation $a_n = a_{n-1} + a_{n-2}$ for all positive integers $n, n \ge 2$. Also, $a_1 = C(1, 0) = 1$ and $a_2 = C(2, 0) + C(1, 1) = 2$, which are f_2 and f_3. It follows that $a_n = f_{n+1}$ for all positive integers n. **19.** $a_n = (n^2 + 3n + 5)(-1)^n$ **21.** $(a_{1,0} + a_{1,1}n + a_{1,2}n^2 + a_{1,3}n^3) + (a_{2,0} + a_{2,1}n + a_{2,2}n^2)(-2)^n + (a_{3,0} + a_{3,1}n)3^n + a_{4,0}(-4)^n$ **23. a)** $3a_{n-1} + 2^n = 3(-2)^n + 2^n = 2^n(-3 + 1) = -2^{n+1} = a_n$ **b)** $a_n = \alpha 3^n - 2^{n+1}$ **c)** $a_n = 3^{n+1} - 2^{n+1}$ **25. a)** $A = -1, B = -7$ **b)** $a_n = \alpha 2^n - n - 7$ **c)** $a_n = 11 \cdot 2^n - n - 7$ **27. a)** $p_3n^3 + p_2n^2 + p_1n + p_0$ **b)** $n^2p_0(-2)^n$ **c)** $n^2(p_1n + p_0)2^n$ **d)** $(p_2n^2 + p_1n + p_0)4^n$ **e)** $n^2(p_2n^2 + p_1n + p_0)(-2)^n$ **f)** $n^2(p_4n^4 + p_3n^3 + p_2n^2 + p_1n + p_0)2^n$ **g)** p_0 **29. a)** $a_n = \alpha 2^n + 3^{n+1}$ **b)** $a_n = -2 \cdot 2^n + 3^{n+1}$ **31.** $a_n = \alpha 2^n + \beta 3^n - n \cdot 2^{n+1} + 3n/2 + 21/4$ **33.** $a_n = (\alpha + \beta n + n^2 + n^3/6)2^n$ **35.** $a_n = -4 \cdot 2^n - n^2/4 - 5n/2 + 1/8 + (39/8)3^n$ **37.** $a_n = n(n+1)(n+2)/6$ **39. a)** $1, -1, i, -i$ **b)** $a_n = \frac{1}{4} - \frac{1}{4}(-1)^n + \frac{2+i}{4}i^n + \frac{2-i}{4}(-i)^n$ **41. a)** Using the formula for f_n, we see that $\left| f_n - \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n \right| = \left| \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \right| < 1/\sqrt{5} < 1/2$. This means that f_n is the integer closest to $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$. **b)** Less when n is even; greater when n is odd **43.** $a_n = f_{n-1} + 2f_n - 1$ **45. a)** $a_n = 3a_{n-1} + 4a_{n-2}$, $a_0 = 2$, $a_1 = 6$ **b)** $a_n = [4^{n+1} + (-1)^n]/5$ **47. a)** $a_n = 2a_{n+1} + (n-1)10{,}000$ **b)** $a_n = 70{,}000 \cdot 2^{n-1} - 10{,}000n - 10{,}000$ **49.** $a_n = 5n^2/12 + 13n/12 + 1$ **51.** See Chapter 11, Section 5 in [Ma93]. **53.** $6^n \cdot 4^{n-1}/n$

Section 7.3

1. 14 **3.** The first step is $(1110)_2(1010)_2 = (2^4 + 2^2)(11)_2\,(10)_2 + 2^2[(11)_2 - (10)_2][(10)_2 - (10)_2] + (2^2 + 1)(10)_2 \cdot (10)_2$. The product is $(10001100)_2$. **5.** $C = 50, 665C + 729 = 33{,}979$ **7. a)** 2 **b)** 4 **c)** 7 **9. a)** 79 **b)** 48,829 **c)** 30,517,579 **11.** $O(\log n)$ **13.** $O(n^{\log_3 2})$ **15.** 5 **17. a)** *Basis step:* If the sequence has just one element, then the one person on the list is the winner. *Recursive step:* Divide the list into two parts—the first half and the second half—as equally as possible. Apply the algorithm recursively to each half to come up with at most two names. Then run through the entire list to count the number of occurrences of each of those names to decide which, if either, is the winner. **b)** $O(n \log n)$ **19. a)** $f(n) = f(n/2) + 2$ **b)** $O(\log n)$ **21. a)** 7 **b)** $O(\log n)$ **23. a)**

```
procedure largest sum(a1, ..., an)
best := 0 {empty subsequence has sum 0}
for i := 1 to n
begin
  sum := 0
  for j := i + 1 to n
  begin
    sum := sum + aj
    if sum > best then best := sum
  end
end
{best is the maximum possible sum of numbers
  in the list}
```

b) $O(n^2)$ **c)** We divide the list into a first half and a second half and apply the algorithm recursively to find the largest sum of consecutive terms for each half. The largest sum of consecutive terms in the entire sequence is either one of these two numbers or the sum of a sequence of consecutive terms that crosses the middle of the list. To find the largest possible sum of a sequence of consecutive terms that crosses the middle of the list, we start at the middle and move forward to find the largest possible sum in the second half of the list, and move backward to find the largest possible sum in the first half of the list; the desired sum is the sum of these two quantities. The final answer is then the largest of this sum and the two answers obtained recursively. The base case is that the largest sum of a sequence of one term is the larger of that number and 0. **d)** 11, 9, 14 **e)** $S(n) = 2S(n/2) + n$, $C(n) = 2C(n/2) + n + 2$, $S(1) = 0$, $C(1) = 1$ **f)** $O(n \log n)$, better than $O(n^2)$ **25.** $(1, 6)$ and $(3, 6)$ at distance 2 **27.** The algorithm is essentially the same as the algorithm given in Example 12. The central strip still has width $2d$ but we need to consider just two boxes of size $d \times d$ rather than eight boxes of size $(d/2) \times (d/2)$. The recurrence relation is the same as the recurrence relation in Example 12, except that the coefficient 7 is replaced by 1. **29.** With $k = \log_b n$, it follows that $f(n) = a^k f(1) + \sum_{j=0}^{k-1} a^j c(n/b^j)^d = a^k f(1) + \sum_{j=0}^{k-1} cn^d = a^k f(1) + kcn^d = a^{\log_b n} f(1) + c(\log_b n)n^d = n^{\log_b a} f(1) + cn^d \log_b n = n^d f(1) + cn^d \log_b n$. **31.** Let $k = \log_b n$ where n is a power of b. *Basis step:* If $n = 1$ and $k = 0$, then $c_1n^d + c_2n^{\log_b a} = c_1 + c_2 = b^dc/(b^d - a) + f(1) + b^dc/(a - b^d) = f(1)$. *Inductive step:* Assume true for k, where $n = b^k$. Then for $n = b^{k+1}$, $f(n) = af(n/b) + cn^d = a\{[b^dc/(b^d - a)](n/b)^d + [f(1) + b^dc/(a - b^d)] \cdot (n/b)^{\log_b a}\} + cn^d = b^dc/(b^d - a)n^da/b^d + [f(1) + b^dc/(a - b^d)]n^{\log_b a} + cn^d = n^d[ac/(b^d - a) + c(b^d - a)/(b^d - a)] + [f(1) + b^dc/(a - b^dc)]n^{\log_b a} = [b^dc/(b^d - a)]n^d + [f(1) + b^dc/(a - b^d)]n^{\log_b a}$. **33.** If

$a > b^d$, then $\log_b a > d$, so the second term dominates, giving $O(n^{\log_b a})$. **35.** $O(n^{\log_4 5})$ **37.** $O(n^3)$

Section 7.4

1. $f(x) = 2(x^6 - 1)/(x - 1)$ **3. a)** $f(x) = 2x(1 - x^6)/(1 - x)$ **b)** $x^3/(1 - x)$ **c)** $x/(1 - x^3)$ **d)** $2/(1 - 2x)$ **e)** $(1 + x)^7$ **f)** $2/(1 + x)$ **g)** $[1/(1 - x)] - x^2$ **h)** $x^3/(1 - x)^2$ **5. a)** $5/(1 - x)$ **b)** $1/(1 - 3x)$ **c)** $2x^3/(1 - x)$ **d)** $(3 - x)/(1 - x)^2$ **e)** $(1 + x)^8$ **f)** $1/(1 - x)^5$ **7. a)** $a_0 = -64$, $a_1 = 144$, $a_2 = -108$, $a_3 = 27$, and $a_n = 0$ for all $n \geq 4$ **b)** The only nonzero coefficients are $a_0 = 1$, $a_3 = 3$, $a_6 = 3$, $a_9 = 1$. **c)** $a_n = 5^n$ **d)** $a_n = (-3)^{n-3}$ for $n \geq 3$, and $a_0 = a_1 = a_2 = 0$ **e)** $a_0 = 8$, $a_1 = 3$, $a_2 = 2$, $a_n = 0$ for odd n greater than 2 and $a_n = 1$ for even n greater than 2 **f)** $a_n = 1$ if n is a positive multiple 4, $a_n = -1$ if $n < 4$, and $a_n = 0$ otherwise **g)** $a_n = n - 1$ for $n \geq 2$ and $a_0 = a_1 = 0$ **h)** $a_n = 2^{n+1}/n!$ **9. a)** 6 **b)** 3 **c)** 9 **d)** 0 **e)** 5 **11. a)** 1024 **b)** 11 **c)** 66 **d)** 292,864 **e)** 20,412 **13.** 10 **15.** 50 **17.** 20 **19.** $f(x) = 1/[(1 - x)(1 - x^2)(1 - x^5)(1 - x^{10})]$ **21.** 15 **23. a)** $x^4(1 + x + x^2 + x^3)^2/(1 - x)$ **b)** 6 **25. a)** The coefficient of x^r in the power series expansion of $1/[(1 - x^3)(1 - x^4)(1 - x^{20})]$ **b)** $1/(1 - x^3 - x^4 - x^{20})$ **c)** 7 **d)** 3224 **27. a)** 3 **b)** 29 **c)** 29 **d)** 242 **29. a)** 10 **b)** 49 **c)** 2 **d)** 4 **31. a)** $G(x) - a_0 - a_1x - a_2x^2$ **b)** $G(x^2)$ **c)** $x^4G(x)$ **d)** $G(2x)$ **e)** $\int_0^x G(t)dt$ **f)** $G(x)/(1 - x)$ **33.** $a_k = 2 \cdot 3^k - 1$ **35.** $a_k = 18 \cdot 3^k - 12 \cdot 2^k$ **37.** $a_k = k^2 + 8k + 20 + (6k - 18)2^k$ **39.** Let $G(x) = \sum_{k=0}^{\infty} f_k x^k$. After shifting indices of summation and adding series, we see that $G(x) - xG(x) - x^2G(x) = f_0 + (f_1 - f_0)x + \sum_{k=2}^{\infty}(f_k - f_{k-1} - f_{k-2})x^k = 0 + x + \sum_{k=2}^{\infty} 0x^k$. Hence, $G(x) - xG(x) - x^2G(x) = x$. Solving for $G(x)$ gives $G(x) = x/(1 - x - x^2)$. By the method of partial fractions, it can be shown that $x/(1 - x - x^2) = (1/\sqrt{5})[1/(1 - \alpha x) - 1/(1 - \beta x)]$, where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. Using the fact that $1/(1 - \alpha x) = \sum_{k=0}^{\infty} \alpha^k x^k$, it follows that $G(x) = (1/\sqrt{5}) \cdot \sum_{k=0}^{\infty}(\alpha^k - \beta^k)x^k$. Hence, $f_k = (1/\sqrt{5}) \cdot (\alpha^k - \beta^k)$. **41. a)** Let $G(x) = \sum_{n=0}^{\infty} C_n x^n$ be the generating function for $\{C_n\}$. Then $G(x)^2 = \sum_{n=0}^{\infty}(\sum_{k=0}^{n} C_k C_{n-k})x^n = \sum_{n=1}^{\infty}(\sum_{k=0}^{n-1} C_k C_{n-1-k})x^{n-1} = \sum_{n=1}^{\infty} C_n x^{n-1}$. Hence, $xG(x)^2 = \sum_{n=1}^{\infty} C_n x^n$, which implies that $xG(x)^2 - G(x) + 1 = 0$. Applying the quadratic formula shows that $G(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$. We choose the minus sign in this formula because the choice of the plus sign leads to a division by zero. **b)** By Exercise 40, $(1 - 4x)^{-1/2} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n$. Integrating term by term (which is valid by a theorem from calculus) shows that $\int_0^x (1 - 4t)^{-1/2}dt = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n}x^{n+1} = x\sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n}x^n$. Because $\int_0^x (1 - 4t)^{-1/2}dt = \frac{1-\sqrt{1-4x}}{2} = xG(x)$, equating coefficients shows that $C_n = \frac{1}{n+1}\binom{2n}{n}$. **43.** Applying the Binomial Theorem to the equality $(1 + x)^{m+n} = (1 + x)^m(1 + x)^n$, shows that $\sum_{r=0}^{m+n} C(m + n, r)x^r = \sum_{r=0}^{m} C(m, r)x^r \cdot \sum_{r=0} C(n, r)x^r = \sum_{r=0}^{m+n} \left[\sum_{k=0}^{r} C(m, r - k)C(n, k)\right] x^r$. Comparing coefficients gives the desired identity. **45. a)** $2e^x$ **b)** e^{-x} **c)** e^{3x} **d)** $xe^x + e^x$ **e)** $(e^x - 1)/x$ **47. a)** $a_n = (-1)^n$ **b)** $a_n = 3 \cdot 2^n$ **c)** $a_n = 3^n - 3 \cdot 2^n$ **d)** $a_n = (-2)^n$ for $n \geq 2$, $a_1 = -3$, $a_0 = 2$ **e)** $a_n = (-2)^n + n!$ **f)** $a_n = (-3)^n + n! \cdot 2^n$ for $n \geq 2$, $a_0 = 1$, $a_1 = -2$ **g)** $a_n = 0$ if n is odd and $a_n = n!/(n/2)!$ if n is even **49. a)** $a_n = 6a_{n-1} + 8^{n-1}$ for $n \geq 1$, $a_0 = 1$ **b)** The general solution of the associated linear homogeneous recurrence relation is $a_n^{(h)} = \alpha 6^n$. A particular solution is $a_n^{(p)} = \frac{1}{2} \cdot 8^n$. Hence, the general solution is $a_n = \alpha 6^n + \frac{1}{2} \cdot 8^n$. Using the initial condition, it follows that $\alpha = \frac{1}{2}$. Hence, $a_n = (6^n + 8^n)/2$. **c)** Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$. Using the recurrence relation for $\{a_k\}$, it can be shown that $G(x) - 6xG(x) = (1 - 7x)/(1 - 8x)$. Hence, $G(x) = (1 - 7x)/[(1 - 6x)(1 - 8x)]$. Using partial fractions, it follows that $G(x) = (1/2)/(1 - 6x) + (1/2)/(1 - 8x)$. With the help of Table 1, it follows that $a_n = (6^n + 8^n)/2$. **51.** $\frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots$ **53.** $(1 + x)(1 + x)^2(1 + x)^3 \cdots$ **55.** The generating functions obtained in Exercises 52 and 53 are equal because $(1 + x)(1 + x^2)(1 + x^3) \cdots = \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1-x^6}{1-x^3} \cdots = \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots$. **57. a)** $G_X(1) = \sum_{k=0}^{\infty} p(X = k) \cdot 1^k = \sum_{k=0}^{\infty} P(X = k) = 1$ **b)** $G'_X(1) = \frac{d}{dx}\sum_{k=0}^{\infty} p(X = k) \cdot x^k|_{x=1} = \sum_{k=0}^{\infty} p(X = k) \cdot k \cdot x^{k-1}|_{x=1} = \sum_{k=0}^{\infty} p(X = k) \cdot k = E(X)$ **c)** $G''_X(1) = \frac{d^2}{dx^2}\sum_{k=0}^{\infty} p(X = k) \cdot x^k|_{x=1} = \sum_{k=0}^{\infty} p(X = k) \cdot k(k - 1) \cdot x^{k-2}|_{x=1} = \sum_{k=0}^{\infty} p(X = k) \cdot (k^2 - k) = V(X) + E(X)^2 - E(X)$. Combining this with part (b) gives the desired results. **59. a)** $G(x) = p^m/(1 - qx)^m$ **b)** $V(x) = mq/p^2$

Section 7.5

1. a) 30 **b)** 29 **c)** 24 **d)** 18 **3.** 1% **5. a)** 300 **b)** 150 **c)** 175 **d)** 100 **7.** 492 **9.** 974 **11.** 55 **13.** 248 **15.** 50,138 **17.** 234 **19.** $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| = |A_1| + |A_2| + |A_3| + |A_4| + |A_5| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_1 \cap A_5| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_2 \cap A_5| - |A_3 \cap A_4| - |A_3 \cap A_5| - |A_4 \cap A_5| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_2 \cap A_5| + |A_1 \cap A_3 \cap A_4| + |A_1 \cap A_3 \cap A_5| + |A_1 \cap A_4 \cap A_5| + |A_2 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_5| + |A_2 \cap A_4 \cap A_5| + |A_3 \cap A_4 \cap A_5| - |A_1 \cap A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_5| - |A_1 \cap A_2 \cap A_4 \cap A_5| - |A_1 \cap A_3 \cap A_4 \cap A_5| - |A_2 \cap A_3 \cap A_4 \cap A_5| + |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5|$ **21.** $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6| = |A_1| + |A_2| + |A_3| + |A_4| + |A_5| + |A_6| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_1 \cap A_5| - |A_1 \cap A_6| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_2 \cap A_5| - |A_2 \cap A_6| - |A_3 \cap A_4| - |A_3 \cap A_5| - |A_3 \cap A_6| - |A_4 \cap A_5| - |A_4 \cap A_6| - |A_5 \cap A_6|$ **23.** $p(E_1 \cup E_2 \cup E_3) = p(E_1) + p(E_2) + p(E_3) - p(E_1 \cap E_2) - p(E_1 \cap E_3) - p(E_2 \cap E_3) + p(E_1 \cap E_2 \cap E_3)$ **25.** 4972/71,295 **27.** $p(E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5) = p(E_1) + p(E_2) + p(E_3) + p(E_4) + p(E_5) - p(E_1 \cap E_2) - p(E_1 \cap E_3) - p(E_1 \cap E_4) - p(E_1 \cap E_5) - p(E_2 \cap E_3) - p(E_2 \cap E_4) - p(E_2 \cap E_5) - p(E_3 \cap E_4) - p(E_3 \cap E_5) - p(E_4 \cap E_5) + p(E_1 \cap E_2 \cap E_3) + p(E_1 \cap E_2 \cap E_4) + p(E_1 \cap E_2 \cap E_5) + p(E_1 \cap E_3 \cap E_4) + p(E_1 \cap E_3 \cap E_5) + p(E_1 \cap E_4 \cap E_5) + p(E_2 \cap E_3 \cap E_4) + P(E_2 \cap E_3 \cap E_5) + p(E_2 \cap E_4 \cap E_5) + p(E_3 \cap E_4 \cap E_5)$ **29.** $p\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{1 \leq i \leq n} p(E_i) -$

$\sum_{1\le i<j\le n} p(E_i \cap E_j) + \sum_{1\le i<j<k\le n} p(E_i \cap E_j \cap E_k) - \cdots + (-1)^{n+1} p\left(\bigcap_{i=1}^{n} E_i\right)$

Section 7.6

1. 75 **3.** 6 **5.** 46 **7.** 9875 **9.** 540 **11.** 2100 **13.** 1854 **15. a)** $D_{100}/100!$ **b)** $100D_{99}/100!$ **c)** $C(100,2)/100!$ **d)** 0 **e)** $1/100!$ **17.** 2,170,680 **19.** By Exercise 18 we have $D_n - nD_{n-1} = -[D_{n-1} - (n-1)D_{n-2}]$. Iterating, we have $D_n - nD_{n-1} = -[D_{n-1} - (n-1)D_{n-2}] = -[-(D_{n-2} - (n-2)D_{n-3})] = D_{n-2} - (n-2)D_{n-3} = \cdots = (-1)^n(D_2 - 2D_1) = (-1)^n$ because $D_2 = 1$ and $D_1 = 0$. **21.** When n is odd **23.** $\phi(n) = n - \sum_{i=1}^{m} \frac{n}{p_i} + \sum_{1\le i<j\le m} \frac{n}{p_i p_j} - \cdots \pm \frac{n}{p_1 p_2 \cdots p_m} = n \prod_{i=1}^{m}\left(a - \frac{1}{p_i}\right)$ **25.** 4 **27.** There are n^m functions from a set with m elements to a set with n elements, $C(n,1)(n-1)^m$ functions from a set with m elements to a set with n elements that miss exactly one element, $C(n,2)(n-2)^m$ functions from a set with m elements to a set with n elements that miss exactly two elements, and so on, with $C(n, n-1) \cdot 1^m$ functions from a set with m elements to a set with n elements that miss exactly $n-1$ elements. Hence, by the principle of inclusion–exclusion, there are $n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \cdots + (-1)^{n-1}C(n,n-1) \cdot 1^m$ onto functions.

Supplementary Exercises

1. a) $A_n = 4A_{n-1}$ **b)** $A_1 = 40$ **c)** $A_n = 10 \cdot 4^n$ **3. a)** $M_n = M_{n-1} + 160{,}000$ **b)** $M_1 = 186{,}000$ **c)** $M_n = 160{,}000n + 26{,}000$ **d)** $T_n = T_{n-1} + 160{,}000n + 26{,}000$ **e)** $T_n = 80{,}000n^2 + 106{,}000n$ **5. a)** $a_n = a_{n-2} + a_{n-3}$ **b)** $a_1 = 0, a_2 = 1, a_3 = 1$ **c)** $a_{12} = 12$ **7. a)** 2 **b)** 5 **c)** 8 **d)** 16 **9.** $a_n = 2^n$ **11.** $a_n = 2 + 4n/3 + n^2/2 + n^3/6$ **13.** $a_n = a_{n-2} + a_{n-3}$ **15.** $O(n^4)$ **17.** $O(n)$ **19. a)** $18n + 18$ **b)** 18 **c)** 0 **21.** $\Delta(a_n b_n) = a_{n+1}b_{n+1} - a_n b_n = a_{n+1}(b_{n+1} - b_n) + b_n(a_{n+1} - a_n) = a_{n+1}\Delta b_n + b_n \Delta a_n$ **23. a)** Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$. Then $G'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$. Therefore, $G'(x) - G'(x) = \sum_{n=0}^{\infty}[(n+1)a_{n+1} - a_n]x^n = \sum_{n=0} x^n/n! = e^x$, as desired. That $G(0) = a_0 = 1$ is given. **b)** We have $[e^{-x}G(x)]' = e^{-x}G'(x) - e^{-x}G(x) = e^{-x}[G'(x) - G(x)] = e^{-x} \cdot e^x = 1$. Hence, $e^{-x}G(x) = x + c$, where c is a constant. Consequently, $G(x) = xe^x + ce^x$. Because $G(0) = 1$, it follows that $c = 1$. **c)** We have $G(x) = \sum_{n=0}^{\infty} x^{n+1}/n! + \sum_{n=0}^{\infty} x^n/n! = \sum_{n=1}^{\infty} x^n/(n-1)! + \sum_{n=0}^{\infty} x^n/n!$. Therefore, $a_n = 1/(n-1)! + 1/n!$ for all $n \ge 1$, and $a_0 = 1$. **25.** 7 **27.** 110 **29.** 0 **31. a)** 19 **b)** 65 **c)** 122 **d)** 167 **e)** 168 **33.** $D_{n-1}/(n-1)!$ **35.** 11/32

CHAPTER 8

Section 8.1

1. a) $\{(0,0),(1,1),(2,2),(3,3)\}$ **b)** $\{(1,3),(2,2),(3,1),(4,0)\}$ **c)** $\{(1,0),(2,0),(2,1),(3,0),(3,1),(3,2),(4,0),(4,1),(4,2),(4,3)\}$ **d)** $\{(1,0),(1,1),(1,2),(1,3),(2,0),(2,2),(3,0),(3,3),(4,0)\}$ **e)** $\{(0,1),(1,0),(1,1),(1,2),(1,3),(2,1),(2,3),(3,1),(3,2),(4,1),(4,3)\}$ **f)** $\{(1,2),(2,1),(2,2)\}$ **3. a)** Transitive **b)** Reflexive, symmetric, transitive **c)** Symmetric **d)** Antisymmetric **e)** Reflexive, symmetric, antisymmetric, transitive **f)** None of these properties **5. a)** Reflexive, transitive **b)** Symmetric **c)** Symmetric **d)** Symmetric **7. a)** Symmetric **b)** Symmetric, transitive **c)** Symmetric **d)** Reflexive, symmetric, transitive **e)** Reflexive, transitive **f)** Reflexive, symmetric, transitive **g)** Antisymmetric **h)** Antisymmetric, transitive **9.** (c), (d), (f) **11. a)** Not irreflexive **b)** Not irreflexive **c)** Not irreflexive **d)** Not irreflexive **13.** Yes, for instance $\{(1,1)\}$ on $\{1,2\}$ **15.** $(a,b) \in R$ if and only if a is taller than b **17.** (a) **19.** None **21.** $\forall a \forall b\,[(a,b) \in R \to (b,a) \notin R]$ **23.** 2^{mn} **25. a)** $\{(a,b) \mid b \text{ divides } a\}$ **b)** $\{(a,b) \mid a \text{ does not divide } b\}$ **27.** The graph of f^{-1} **29. a)** $\{(a,b) \mid a$ is required to read or has read $b\}$ **b)** $\{(a,b) \mid a$ is required to read and has read $b\}$ **c)** $\{(a,b) \mid$ either a is required to read b but has not read it or a has read b but is not required to$\}$ **d)** $\{(a,b) \mid a$ is required to read b but has not read it$\}$ **e)** $\{(a,b) \mid a$ has read b but is not required to$\}$ **31.** $S \circ R = \{(a,b) \mid a$ is a parent of b and b has a sibling$\}$, $R \circ S = \{(a,b) \mid a$ is an aunt or uncle of $b\}$ **33. a)** $\mathbf{R}^2$ **b)** R_6 **c)** R_3 **d)** R_3 **e)** $\emptyset$ **f)** R_1 **g)** R_4 **h)** R_4 **35. a)** R_1 **b)** R_2 **c)** R_3 **d)** $\mathbf{R}^2$ **e)** R_3 **f)** $\mathbf{R}^2$ **g)** $\mathbf{R}^2$ **h)** $\mathbf{R}^2$ **37.** b got his or her doctorate under someone who got his or her doctorate under a; there is a sequence of $n+1$ people, starting with a and ending with b, such that each is the advisor of the next person in the sequence **39. a)** $\{(a,b) \mid a - b \equiv 0, 3, 4, 6, 8, \text{ or } 9 \pmod{12}\}$ **b)** $\{(a,b) \mid a \equiv b \pmod{12}\}$ **c)** $\{(a,b) \mid a - b \equiv 3, 6, \text{ or } 9 \pmod{12}\}$ **d)** $\{(a,b) \mid a - b \equiv 4 \text{ or } 8 \pmod{12}\}$ **e)** $\{(a,b) \mid a - b \equiv 3, 4, 6, 8, \text{ or } 9 \pmod{12}\}$ **41.** 8 **43. a)** 65,536 **b)** 32,768 **45. a)** $2^{n(n+1)/2}$ **b)** $2^n 3^{n(n-1)/2}$ **c)** $3^{n(n-1)/2}$ **d)** $2^{n(n-1)}$ **e)** $2^{n(n-1)/2}$ **f)** $2^{n^2} - 2 \cdot 2^{n(n-1)}$ **47.** There may be no such b. **49.** If R is symmetric and $(a,b) \in R$, then $(b,a) \in R$, so $(a,b) \in R^{-1}$. Hence, $R \subseteq R^{-1}$. Similarly, $R^{-1} \subseteq R$. So $R = R^{-1}$. Conversely, if $R = R^{-1}$ and $(a,b) \in R$, then $(a,b) \in R^{-1}$, so $(b,a) \in R$. Thus R is symmetric. **51.** R is reflexive if and only if $(a,a) \in R$ for all $a \in A$ if and only if $(a,a) \in R^{-1}$ [because $(a,a) \in R$ if and only if $(a,a) \in R^{-1}$] if and only if R^{-1} is reflexive. **53.** Use mathematical induction. The result is trivial for $n = 1$. Assume R^n is reflexive and transitive. By Theorem 1, $R^{n+1} \subseteq R$. To see that $R \subseteq R^{n+1} = R^n \circ R$, let $(a,b) \in R$. By the inductive hypothesis, $R^n = R$ and hence, is reflexive. Thus $(b,b) \in R^n$. Therefore $(a,b) \in R^{n+1}$. **55.** Use mathematical induction. The result is trivial for $n = 1$. Assume R^n is reflexive. Then $(a,a) \in R^n$ for all $a \in A$ and $(a,a) \in R$. Thus $(a,a) \in R^n \circ R = R^{n+1}$ for all $a \in A$. **57.** No, for instance, take $R = \{(1,2),(2,1)\}$.

Section 8.2

1. $\{(1,2,3),(1,2,4),(1,3,4),(2,3,4)\}$ **3.** (Nadir, 122, 34, Detroit, 08:10), (Acme, 221, 22, Denver, 08:17), (Acme, 122,